Math 210C Lecture 13 Notes

Daniel Raban

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1 The Five Lemma, Chain Complexes, and Homology

1.1 The five lemma

I am omitting Professor Shatrifi's proof of the Snake Lemma. It's tough to read in notes and also tough to write down in real-time.[1] The following lemma has the same sort of argument, as well.

Lemma 1.1 (Five lemma). *Let*

$$\begin{array}{ccccccccc} A & \xrightarrow{e} & B & \xrightarrow{f} & C & \xrightarrow{g} & D & \xrightarrow{h} & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\ A' & \xrightarrow{e'} & B' & \xrightarrow{f'} & C' & \xrightarrow{g'} & D' & \xrightarrow{h'} & E \end{array}$$

be a morphism of exact sequences in an abelian category $\mathcal{C}$.

1. *If β and δ are epimorphisms and ε is a monomorphism, then γ is an epimorphism.*
2. *If β and δ are monomorphisms and α is an epimorphism, then γ is a monomorphism.*
3. *If β and δ are isomorphisms, ε is a monomorphism, and α is an epimorphism, then γ is an isomorphism.*

Proof. We will prove the first statement in the category of R-modules. Let $c' \in C'$. Then there is a $d \in D$ such that $\delta(d) = g'(c')$ (as g' is an epimorphism). Then $\varepsilon(h(d)) = h'(\delta(d)) = h'(g'(c')) = 0$. Since ε is a monomorphism, $h(d) = 0$. So $d = g(c)$ for some $c \in C$. So $g'(c' - \gamma(c)) = g'(c') - \delta(g(c)) = g'(c') - \delta(d) = 0$. So there exists a $b' \in B'$ such that $f'(b') = c' - \gamma(c)$. β is an epimorphism, so there is a $b \in B$ such that $\beta(b) = b'$. Now $\gamma(f(b)) = f'(\beta(b)) = f'(b') = c' - \gamma(c)$, so $c' = \gamma(f(b)) + \gamma(c) = \gamma(f(b) + c)$. That is, $c' \in \operatorname{im}(\gamma$, which means that γ is surjective. $\square$

[1] I have a proof of the Snake Lemma in my UC Berkeley Math 250A notes.

If $f : A \to B$ is a morphism, we have the 4-term exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow A \longrightarrow B \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

We can turn this into two 3-term exact sequences:

$$0 \longrightarrow \ker(f) \longrightarrow A \longrightarrow \operatorname{coim}(f) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im}(f) \longrightarrow B \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

We can always do this is a 4-term exact sequence.

1.2 Chain complexes and homology

Definition 1.1. A (**chain**) **complex** in an abelian category $\mathcal{C}$ is a sequence $A_\cdot = (A_i, A_i^A)_{i\in\mathbb{Z}}$ with defining interval $\mathbb{Z}$ such that $d_i^A \circ d_{i+1}^A = 0$:

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}^A} A_i \xrightarrow{d_i^A} A_{i-1} \longrightarrow \cdots$$

where $\operatorname{im}(d_{i+1}^A) \to \ker(d_i^A)$ is a monomorphism for all I. A morphsim of complex is a morphism of sequences.

Remark 1.1. A "complex" $(A_i)_{i\geq 0}$ is a complex where $A_i = 0$ for all $i < 0$.

Definition 1.2. The i-th **homology group** of a complex $A_\cdot$ is $H_i(A) = \ker(d_i^A)/\operatorname{im}(d_{i+1}^A)$.

Definition 1.3. A **cochain complex** is a sequence $A^\cdot = (A^i, d_A^i)_{i\in\mathbb{Z}}$ with $d_A^i : A^i \to A^{i+1}$

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A_i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \cdots$$

Definition 1.4. The i-th **cohomology group** of a cochain complex $A^\cdot$ is $H^i(A) = \ker(d_A^i)/\operatorname{im}(d_A^i i + 1)$.

Remark 1.2. A is exact iff $H_i(A) = 0$ for al $i \in \mathbb{Z}$.

The homology groups measure how close your sequence is to being exact.

Example 1.1. Consider the chain complex in Ab:

$$\cdots \longrightarrow \mathbb{Z}/n^3\mathbb{Z} \xrightarrow{n^2} \mathbb{Z}/n^3\mathbb{Z} \xrightarrow{n^2} \mathbb{Z}/n^3\mathbb{Z} \longrightarrow \cdots$$

Then

$$H_i(A) = \frac{n\mathbb{Z}/n^3\mathbb{Z}}{n^2\mathbb{Z}/n^3\mathbb{Z}} \cong n\mathbb{Z}/n^2\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

Lemma 1.2. *Let $f_. : A_. \to B_.$ be a morphism of complexes. Then there is a morphism $f_i^* : H_i(A) \to H_i(B)$, natural in f such that the diagram*

$$\begin{tikzcd}
\ker(d_i^A) \arrow[r] \arrow[d] & A_i \arrow[r, "f_i"] & B_i \arrow[r] & \operatorname{coker}(d_{i+1}^B) \\
H_i(A) \arrow[rrr, "f_i^*"] & & & H_i(B) \arrow[u]
\end{tikzcd}$$

commutes.

Proof. We will prove this in $\mathcal{C} = R - \mathrm{Mod}$. Let $a \in \ker(d_i^A)$ maps to $\overline{a} \in H_i(A)$. Then $d - i^B(f_i(A) = f_{i-1}(d_i^A(a)) = 0$. So $f_i(a) \in ker(d_i^B)$. So $\ker(d_i^A \to \ker(d_i^B)$, If $a = d_{i+1}^A$, then $f_i(a) = d_{i+1}^B(d_{i+1}(a')) \in \mathrm{im}(d_{i+1}^B)$, and the map factors through $H_i(A)$ to give f_i^*. Check that $f_i^*(A) = \overline{f_i(a)}$. □